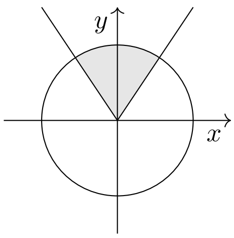
3801. The region R_k is shaded below:



The gradient of the mod graph is $\pm k$. So, the angle between the mod graph and the x axis is $\arctan k$. Hence, the angle subtended at the origin by the shaded sector is $\pi - 2\arctan k$. Since $r = 1$, this is also the arc length. The perimeter P_k of region R_k is then two radii of length 1 plus an arc. This gives $P_k = 2 + \pi - 2\arctan k$, as required.

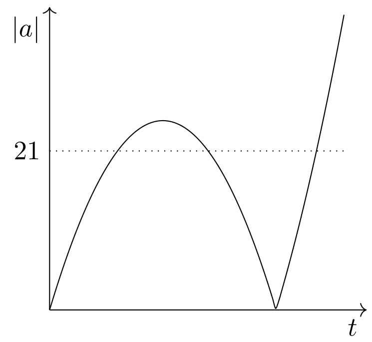
3802. Reflection in $y = x + k$ can be seen as reflection in the line $y = x$ following by a translation. The origin is transformed to the point $(-k, k)$, so the translation has vector $-k\mathbf{i} + k\mathbf{j}$.

The curve $y = x^2$, therefore, is reflected to $x = y^2$, and then translated to $x + k = (y - k)^2$.

3803. No implication links these.

- A counterexample to forwards implication: $\mathrm{f}(x) = (x-1)^4$, which has a factor of $(x-1)^3$ and a local minimum at $x = 1$.
- A counterexample to backwards implication: $\mathrm{f}(x) = (x-1)^3 + 2$, with a point of inflection at $x = 1$ but no factor of $(x-1)^3$.

3804. The acceleration is $a = \frac{F}{m} = 10t - t^2$. Sketching $|a|$ against t, this is



So, we solve

$$\begin{aligned} 10t - t^2 = 21 &\implies t = 3, 7, \\ 10t - t^2 = -21 &\implies t = 5 \pm \sqrt{46}. \end{aligned}$$

The negative root isn't relevant. Therefore, the magnitude of the acceleration is greater than 21 ms^{-2} for $t \in (3, 7) \cup (5 + \sqrt{46}, \infty)$.

3805. We look for a and b such that

$$4y^4 - 8y^2 \equiv a(by)^4 - a(by)^2.$$

Equating coefficients,

$$\begin{aligned} ab^4 &= 4, \\ ab^2 &= 8. \end{aligned}$$

Dividing these gives $b^2 = \frac{1}{2}$. So, $b = 1/\sqrt{2}$ and $a = 16$. The transformation, therefore, is a stretch factor 16 in the x direction and a stretch factor $\sqrt{2}$ in the y direction. The combined area scale factor is $16\sqrt{2}$.

3806. Place the $m \times n$ lattice in the positive quadrant, with the lower-left corner at $(0, 0)$ and the upper-right at (m, n). Assume, for a contradiction, that the line $y = \frac{n}{m}x$ does not pass through any lattice points other than $(0, 0)$ and (m, n), and also that $\mathrm{hcf}(m, n) = k$, where $k > 1$.

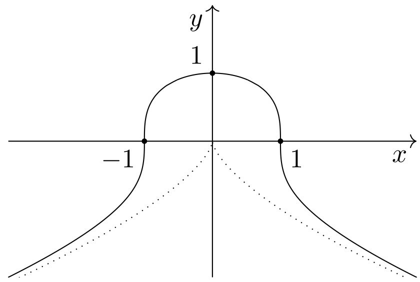
We can write $m = ka$ and $n = kb$, for integers $a, b > 1$. The point (a, b) is a lattice point. And, since the gradient $\frac{b}{a} = \frac{kb}{ka} = \frac{n}{m}$, the point (a, b) lies on $y = \frac{n}{m}x$. This is a contradiction. Hence, if an $m \times n$ lattice is such that its diagonals do not pass through any lattice points, then $\mathrm{hcf}(m, n) = 1$. □

3807. (a) Differentiating implicitly,

$$2x + 3y^2 \frac{dy}{dx} = 0.$$

The points are $(\pm 1, 0)$ and $(0, 1)$. Subbing in gives tangents parallel to the y axis at $(\pm 1, 0)$ and parallel to the x axis at $(0, 1)$. These have equations $x = \pm 1$ and $y = 1$.

(b) As $x \to \pm\infty$, the curve approaches $x^2 + y^3 = 0$. This is $y = -x^{\frac{2}{3}}$, which is the dotted curve in the diagram. Combined with the tangents and points in (a), the curve is



3808. Firstly, we multiply top and bottom by e^x. Then we let $u = e^x + 2$, so that $du = e^x\,dx$, which gives $dx = \frac{1}{u-2}\,du$. Enacting the substitution,

$$\begin{aligned} &\int \frac{e^x - 2}{e^x + 2}\,dx \\ = &\int \frac{u-4}{u} \cdot \frac{1}{u-2}\,du \\ \equiv &\int \frac{u-4}{u(u-2)}\,du. \end{aligned}$$

Writing in partial fractions, this is

$$\begin{aligned} &\int \frac{2}{u} - \frac{1}{u-2}\,du \\ &= 2\ln|u| - \ln|u-2| + c \\ &= 2\ln(e^x+2) - x + c. \end{aligned}$$

——— ALTERNATIVE METHOD ———

The integrand can be written as

$$\begin{aligned} &\frac{1-2e^{-x}}{1+2e^{-x}} \\ &\equiv \frac{e^x-2}{e^x+2} \\ &\equiv \frac{2e^x-(e^x+2)}{e^x+2} \\ &\equiv \frac{2e^x}{e^x+2} - 1. \end{aligned}$$

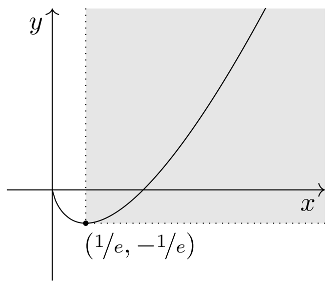
We can now integrate by inspection:

$$\begin{aligned} &\int \frac{2e^x}{e^x+2} - 1\,dx \\ &\equiv 2\ln(e^x+2) - x + c. \end{aligned}$$

3809. By the product rule, the derivative is

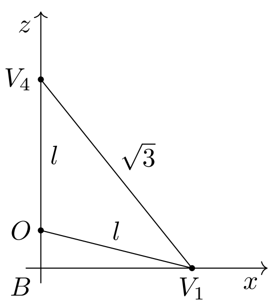
$$f'(x) = \ln x + 1.$$

For SPs, $\ln x = -1$, so $x = 1/e$. Hence, the curve $y = x\ln x$ is stationary at $(1/e, -1/e)$. Sketching this, the graph is



The largest domain and codomain over which $f(x) = x\ln x$ is invertible is shaded above. The domain is $[1/e, \infty)$ and the codomain is $[-1/e, \infty)$.

3810. Place the three base vertices V_1, V_2, V_3 on an (x, y) plane at $(\cos\theta, \sin\theta)$, for $\theta = 0, 120, 240°$, forming an equilateral triangle with side length $\sqrt{3}$. The distance from the centre of the triangle B to each vertex is 1. The fourth vertex is then on the z axis. The (x, z) plane is as follows:



Since $|V_1V_4| = \sqrt{3}$, the z coordinate of V_4 is $\sqrt{2}$. Let l be length $|OV_1| = |OV_4|$. Pythagoras on $\triangle OBV_1$ then gives

$$\begin{aligned} &(\sqrt{2} - l)^2 + 1^2 = l^2 \\ \implies &l = \tfrac{3}{2\sqrt{2}}. \end{aligned}$$

Using the cosine rule on $\triangle OV_1V_4$, we get

$$\angle V_1OV_4 = \arccos\left(\tfrac{1}{3}\right) \approx 109.5°.$$

3811. Setting the first derivative to zero,

$$\begin{aligned} &3x^2 - 3 = 0 \\ \implies &x \pm 1. \end{aligned}$$

So, the local max is at $(-1, 12)$. The equation of the surface of the deepest pool is $y = 12$. This intersects the curve at $x = -1$ and $x = 2$. So, the cross-sectional area of the pool is

$$\begin{aligned} &\int_{-1}^{2} 12 - (10 + x^3 - 3x)\,dx \\ &= \left[-\tfrac{1}{4}x^4 + \tfrac{3}{2}x^2 + 2x\right]_{-1}^{2} \\ &= (6) - (-0.75) \\ &= 6.75 \text{ m}^2, \text{ as required.} \end{aligned}$$

3812. Using the binomial expansion,

$$\begin{aligned} &\lim_{h\to 0} \frac{(2x+h)^n - (2x)^n}{(x+h)^n - x^n} \\ &\equiv \lim_{h\to 0} \frac{(2x)^n + n(2x)^{n-1}h + ... - (2x)^n}{x^n + nx^{n-1}h + ... - x^n} \\ &\equiv \lim_{h\to 0} \frac{n(2x)^{n-1}h + ...}{nx^{n-1}h + ...} \\ &\equiv \lim_{h\to 0} \frac{(2x)^{n-1} + ...}{x^{n-1} + ...} \end{aligned}$$

The other terms (denoted by the ellipses) contain factors of h, so tend to zero when we take the limit. The factors of x^{n-1} on the top and bottom then cancel (since $x \neq 0$), which leaves 2^{n-1}.

3813. The problem is symmetrical in $y = x$: the first two parabolae are reflections of each other in this line and the third is a reflection of itself. The first parabola is tangent to $y = x$, since $x^2 + x = x$ is $x^2 = 0$, which has a double root at $x = 0$. So, the first two parabolae are tangent to each other.

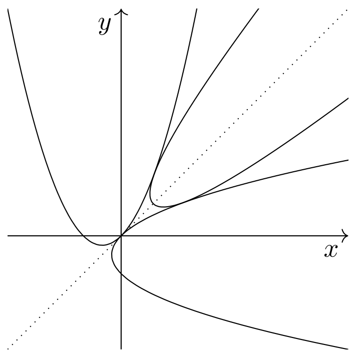
Solving for intersections of the first and third, we rearrange the first to $y - x = x^2$, and substitute into the third:

$$\begin{aligned} &x + (x^2 + x) = x^4 + 2 \\ \implies &x^4 - x^2 - 2x + 2 = 0. \end{aligned}$$

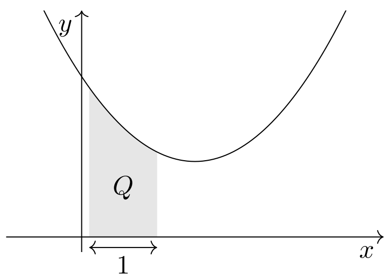
A polynomial solver gives $x = 1$. The quartic duly factorises as

$$\begin{aligned} &x^4 - x^2 - 2x + 2 \\ &\equiv (x-1)^2(x^2 + 2x + 2). \end{aligned}$$

The presence of a double factor indicates that the first and third parabolae are tangent at $(1, 2)$. The symmetry of the problem dictates that the second and third are therefore tangent:



3814. The graph $y = \mathrm{f}(x)$ is a positive parabola which is always above the x axis. The quantity Q is the area of a region below the curve, between $x = p$ and $x = p+1$. This is a strip of width 1, as shown below:



We need to find the vertex. We are told that Q is minimised at $p = 3$. This minimisation occurs when the strip is placed symmetrically beneath the vertex of the parabola. This has the sides of the strip at $x = 2.5$ and $x = 3.5$. Hence, $x = 3$.

3815. There are n horizontals, n verticals, and 2 long diagonals, giving $2n + 2$ ways overall.

3816. The probabilities on each pair of branches must sum to 1. So, $q = \frac{1}{4}$. Also

$$\begin{aligned} &3p^2 + 2p = 1 \\ \Longrightarrow\ &p = -1, \tfrac{1}{3} \end{aligned}$$

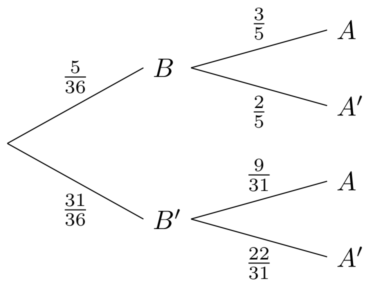
Probabilities must be positive, so $p = \frac{1}{3}$. These values give $\mathbb{P}(B) = \frac{3}{36} + \frac{2}{36}$, so

$$\begin{aligned} \mathbb{P}(A \mid B) &= \tfrac{3}{5}, \\ \mathbb{P}(A' \mid B) &= \tfrac{2}{5}. \end{aligned}$$

Likewise, $\mathbb{P}(B') = \frac{9}{36} + \frac{22}{36}$, so

$$\begin{aligned} \mathbb{P}(A \mid B') &= \tfrac{9}{31}, \\ \mathbb{P}(A' \mid B) &= \tfrac{22}{31}. \end{aligned}$$

The tree diagram conditioned on B is



3817. (a) Rearranging the proposed quadratic gives

$$y/x = ax + b.$$

So, if there is a linear relationship between x and y/x, then the quadratic relationship holds between x and y. Linear relationships are easy to analyse.

(b) The equation of the straight line through $(0, 2.42)$ and $(4, 1.87)$ is

$$y/x = -0.1375x + 2.42.$$

The equation of the associated quadratic is

$$y = -0.1375x^2 + 2.42x.$$

3818. The second equation is $x^2 + y^2 + z = 12$. When we subtract the first equation from this, both x^2 and y^2 cancel, leaving $z^2 + z - 12 = 0$. This gives two possible values, $z = 3, -4$.

— Nota Bene —

Since we have two equations in three unknowns, the variables x and y cannot be found. All we know is that $x^2 + y^2$ is 9 or 16. It is only the particular form of these two equations that allow solving for z; in general, with two equations in three unknowns, no variable is fixed.

3819. Writing algebraically,

$$\begin{aligned} \frac{p^8}{p^8 + 8p^7(1-p)} &= \frac{1}{25} \\ \Longrightarrow 25p^8 &= p^8 + 8p^7(1-p) \\ \Longrightarrow p^7(4p-1) &= 0 \\ \Longrightarrow p &= 0, \tfrac{1}{4}. \end{aligned}$$

The conditional probability $\mathbb{P}(X = 8 \mid X \geq 7)$ is undefined for $p = 0$, so we reject that root. This leaves $p = \frac{1}{4}$.

3820. Differentiating twice, each index is reduced by 2. So, the second derivative has the form

$$f''(x) = b_1x^2 + b_2x^4 + ... + b_{k-1}x^{2k-2},$$

where $b_1, ..., b_{k-1} \in \mathbb{R}$ are new non-zero constants. Clearly $f''(0) = 0$. But, since every index in $f''(x)$ is even, the values of the second derivative are symmetrical around $x = 0$, i.e. $f''(-x) \equiv f''(x)$. Hence, the second derivative does not change sign at the origin, and there is no point of inflection.

3821. There are ${}^8C_3$ sets of three vertices which can be chosen. To form a triangle congruent to the one shown, we need two vertices on an edge and one on the diagonally opposite edge.

There are 12 edges to choose from, and then 2 vertices to choose from on the diagonally opposite edge. So, the probability is

$$p = \frac{12 \times 2}{{}^8C_3} = \tfrac{3}{7}.$$

——— Alternative Method ———

The triangle has lengths $(1, \sqrt{2}, \sqrt{3})$. Choose the first vertex wlog. Then the distance to the next vertex chosen can be 1 with probability $\frac{3}{7}$, $\sqrt{2}$ with probability $\frac{3}{7}$ or $\sqrt{3}$ with probability $\frac{1}{7}$. Counting up, in each case, the number of successful locations for the third vertex, the overall probability is

$$p = \tfrac{3}{7} \cdot \tfrac{2}{6} + \tfrac{3}{7} \cdot \tfrac{2}{6} + \tfrac{1}{7} \cdot \tfrac{6}{6} = \tfrac{3}{7}.$$

3822. For $x \in (1, 2)$, both x and $\ln x$ are positive, so the value $x \ln x$ is positive on this interval. The curve $y = x \ln x$ is above the x axis.

The first derivative is $\ln x + 1$, and the second derivative is $1/x$. This is positive on the interval $(1, 2)$, so $y = x \ln x$ is concave.

So, any chord joining distinct points $x_1, x_2 \in [1, 2]$ on the curve $y = x \ln x$ will be above the curve everywhere except its endpoints. Therefore, each trapezium must overestimate the area beneath the curve. Hence, the rule must do so overall. □

3823. Assuming $p \neq 0$, the line has gradient m given by

$$\frac{(4p^2 - 1) - (2p - 1)}{(p^2 - 1) - (p - 1)} \equiv \frac{4p^2 - 2p}{p^2 - p} \equiv \frac{4p - 2}{p - 1}.$$

We substitute m into $y - y_1 = m(x - x_1)$, with (x_1, y_1) as $(p - 1, 2p - 1)$ and (x, y) as $(3, 0)$:

$$\begin{aligned} &-(2p-1) = \tfrac{4p-2}{p-1}\big(3 - (p-1)\big) \\ \implies &p = \tfrac{1}{2}, 7. \end{aligned}$$

The value $p = \frac{1}{2}$ gives gradient $m = 0$, so the line doesn't *cross* the x axis at $x = 3$, it *is* the x axis. Hence, the answer is $p = 7$.

3824. (a) The full list of divisors is

$$1, d_2, ..., d_n, P.$$

The first divisor is $d_1 = 1$, so $P/d_1 = P$.

(b) The divisors d_2 and d_n must make a pair of factors of P, so

$$\frac{P}{d_2} = d_n.$$

Then $\dfrac{P}{d_3} = d_{n-1}$ etc. The general case is

$$\frac{P}{d_k} = d_{n-k+2}.$$

(c) We take out a factor of $\frac{1}{P}$, to produce terms of the form simplified in (a):

$$\begin{aligned} &\frac{1}{d_1} + \frac{1}{d_2} + ... + \frac{1}{d_n} + \frac{1}{P} \\ &= \frac{1}{P}\left(\frac{P}{d_1} + \frac{P}{d_2} + ... + \frac{P}{d_n} + 1\right) \\ &= \frac{1}{P}(P + d_n + ... + d_2 + d_1) \\ &= \frac{1}{P}(P + P) \\ &= 2, \text{ as required.} \end{aligned}$$

3825. The possibility space consists of all sets of rolls adding to 5. These are three outcomes $(1, 1, 3)$ and three outcomes $(1, 2, 2)$. Outcomes $(1, 1, 3)$ cannot be successful. Two of the outcomes $(1, 2, 2)$ are successful: those with $Z = 2$. Hence,

$$\mathbb{P}(XY = Z \mid X + Y + Z = 5) = \tfrac{2}{6} = \tfrac{1}{3}.$$

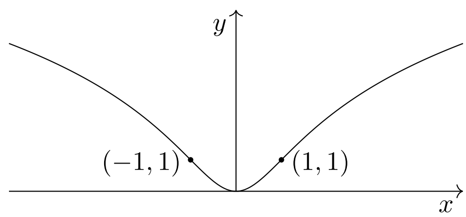
3826. The input $x^2 + 1$ has even symmetry; hence, the curve does too. For x intercepts,

$$\begin{aligned} &\log_2(x^2 + 1) = 0 \\ \implies &x = 0. \end{aligned}$$

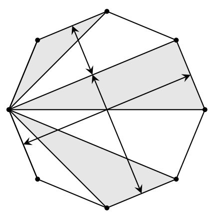
So, the only axis intercept is the origin. Also, $y \geq 0$ for all x. The derivative is

$$\frac{dy}{dx} = \frac{2x}{(x^2 + 1)\ln 2}.$$

So, the only SP is at the origin. Together with the convex/concave information, which gives points of inflection at $(\pm 1, 1)$, the graph is



3827. Let the octagon have unit side length. Then each triangle can be considered as having base 1.



The height of the big triangle is the distance across the octagon, as shown. The heights of the smaller triangles add to the same distance. Therefore, the areas of the smaller two triangles add to the area of the larger. □

3828. For SPs, $5x^4 - 6x + 3 = 0$. The Newton-Raphson iteration for this equation is

$$x_{n+1} = x_n - \frac{5x_n^4 - 6x_n + 3}{20x_n^3 - 6}.$$

Running this twice with different starting points,

x_0	x_1	Limit of x_n
0	0.5	0.63853...
1	$6/7$	0.69941...

Using a sign-change method for proof, defining $\text{g}(x) = 5x^4 - 6x + 3$, we test

$$\begin{aligned}\text{g}(0.6) &= 0.048 > 0,\\ \text{g}(0.65) &= -0.0075 < 0\\ \text{g}(0.7) &= 0.0005 > 0.\end{aligned}$$

So, the curve $y = x^5 - 3x^2 + 3x$ has two stationary points which lie close together, as required.

3829. Compare the ranges, to which the inputs x, x^2, x^3 don't make a difference. On the left, the ranges are $[-1, 1]$. On the right, the modulus function is applied to the outputs, so the ranges are all $[0, 1]$.

(a) False,
(b) False,
(c) False.

3830. (a) Factorising the RHS as a quadratic in $\sqrt{x}$ and a, the family of curves is

$$y = \left(\sqrt{x} - a\right)^2.$$

If $a > 0$, then this has a double root at $\sqrt{x} = a$, which is a point of tangency with the x axis.

(b) Each of the curves C_a is well defined at the y axis. Differentiating,

$$\begin{aligned}& y = \left(\sqrt{x} - a\right)^2\\ \Longrightarrow\ & \frac{dy}{dx} = \frac{\sqrt{x} - a}{\sqrt{x}}.\end{aligned}$$

The denominator is zero at the y axis, so the gradient is undefined. In general, this signifies a tangent parallel to the y axis. At $x = 0$, this signifies that the tangent is the y axis.

3831. (a) The x and y coordinates are $\cos t$ and $\sin t$. Squaring and adding, the first Pythagorean trig identity gives $x^2 + y^2 = 1$. This is a unit circle in the (x, y) plane, appearing at every z value. Hence, it is a cylinder.

(b) Differentiating, the velocity is

$$\mathbf{v} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}.$$

As in (a), the Pythagorean sum of the x and y components is 1. So, the speed is $\sqrt{2}$.

(c) The acceleration vector is

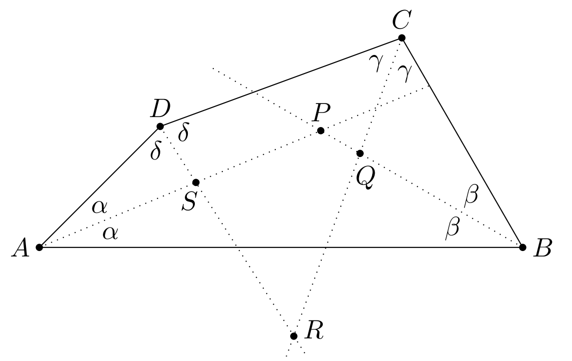
$$\begin{aligned}\mathbf{a} &= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}\\ &\equiv -\big((\cos t)\mathbf{i} + (\sin t)\mathbf{j}\big).\end{aligned}$$

This has no component in z. So, consider only an (x, y) plane. The position (ignoring z) is

$$\mathbf{r}_{(x,y)} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}.$$

The acceleration, then, is $\mathbf{a} = -\mathbf{r}_{(x,y)}$. Since the acceleration is the negative of position, it is directed towards the origin of the (x, y) plane. This is the axis of symmetry of the cylinder.

3832. Call the half-interior angles α at A, β at B and so on. The scenario is as follows:



We know that $\alpha + \beta + \gamma + \delta = 180°$. As well as this, $\angle APB = 180° - \alpha - \beta$ and $\angle CRD = 180° - \gamma - \delta$. Adding these together,

$$\begin{aligned}&\angle APB + \angle CRD\\ &= 180° - \alpha - \beta + 180° - \gamma - \delta\\ &\equiv 360° - (\alpha + \beta + \gamma + \delta)\\ &= 180°.\end{aligned}$$

Hence, opposite angles in $PQRS$ add to 180°. Therefore, $PQRS$ is a cyclic quadrilateral. QED.

3833. This is true. In 2D, an example would be

$$\begin{aligned} x_1 + x_2 &= 0, \\ 2(x_1 + x_2) &= 0. \end{aligned}$$

These are simultaneously satisfied (because they are essentially the same equation) by infinitely many points.

Generalising to n dimensions, set up the following system of n linear equations:

$$n \text{ equations}: \begin{cases} x_1 + x_2 + ... + x_n = 0, \\ 2(x_1 + x_2 + ... + x_n) = 0, \\ ... \\ n(x_1 + x_2 + ... + x_n) = 0. \end{cases}$$

This is a set of n linear equations in n unknowns with infinitely many $(x_1, ..., x_n)$ solution points, proving the result by construction.

3834. Factorising the denominator of the LHS,

$$\begin{aligned} &\frac{1}{\cos^2 x - \cos^4 x} \\ \equiv\ &\frac{1}{\cos^2 x(1 - \cos^2 x)} \\ \equiv\ &\frac{1}{\cos^2 x \sin^2 x}. \end{aligned}$$

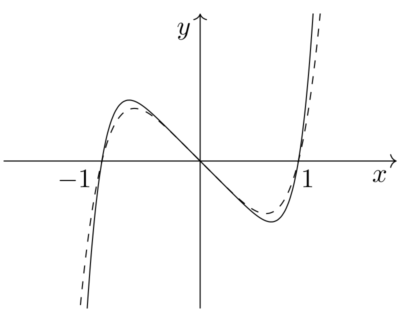
Using $\sin 2\theta \equiv 2 \sin\theta \cos\theta$, this is

$$\begin{aligned} &\frac{4}{4\cos^2 x \sin^2 x} \\ \equiv\ &\frac{4}{\sin^2 2x} \\ \equiv\ &4 \operatorname{cosec}^2 2x, \text{ as required.} \end{aligned}$$

3835. ① Factorising, $y = x(x+1)(x-1)(x^2+1)$. This is a positive quintic with single roots at $x = 0, \pm 1$. The quadratic factor has no roots.

② Factorising, $y = x(x+1)(x-1)(x^4+x^2+1)$. This is a positive septic with single roots at $x = 0, \pm 1$. The quartic factor has $\Delta = -3 < 0$, so has no real roots.

For intersections, $x^5 - x = x^7 - x$, which is $x^5(1-x^2) = 0$. This has roots $x = 0, \pm 1$, which are all x intercepts. The root at $x = 0$ is quintuple, so the curves are tangent at O, crossing each other. Putting all of this together, the curves, with the quintic shown dashed, are



3836. (a) Consider the multiplicity of the roots. We have a single root and a repeated root (stationary point). If this were a double root, then one linear factor would remain, yielding another root. But we are told that there are exactly two roots. Hence, the stationary point must be a triple root, i.e. a point of inflection. So, the second derivative must be zero.

(b) Setting the second derivative to zero,

$$\begin{aligned} & 324x^2 + 324x - 360 = 0 \\ \implies & 9x^2 + 9x - 10 = 0 \\ \implies & (3x-2)(3x+5) = 0 \\ \implies & x = \tfrac{2}{3}, -\tfrac{5}{3}. \end{aligned}$$

(c) Testing the above values, we find that $x = \frac{2}{3}$ is the stationary root, which must be a point of inflection. Hence, $\mathrm{f}(x)$ must have a factor of $(3x-2)^3$. Taking this out,

$$\begin{aligned} & 27x^4 + 54x^3 - 180x^2 + 136x - 32 = 0 \\ \implies & (3x-2)^3(x+4) = 0 \\ \implies & x = \tfrac{2}{3}, -4. \end{aligned}$$

3837. Considering all 11 letters to be distinguishable, there are 11! anagrams. In this list, we overcount by factors of 5! for the As, 2! for the Bs and 2! for the Rs. Hence, the total number of anagrams is

$$\frac{11!}{5!2!2!} = 83160, \text{ as required.}$$

3838. The maximum height is given by

$$\begin{aligned} 0 &= u^2 \sin^2\theta - 2gh \\ \implies h &= \frac{u^2 \sin^2\theta}{2g}. \end{aligned}$$

The time of flight is $t = \frac{2u\sin\theta}{g}$, so the range is

$$\begin{aligned} R &= (u\cos\theta)\frac{2u\sin\theta}{g} \\ &\equiv \frac{2u^2\sin\theta\cos\theta}{g}. \end{aligned}$$

We know that $h = \frac{1}{2}R$, which gives

$$\begin{aligned} & \frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin\theta\cos\theta}{g} \\ \implies & \tfrac{1}{2}\sin\theta = \cos\theta \\ \implies & \tan\theta = 2. \end{aligned}$$

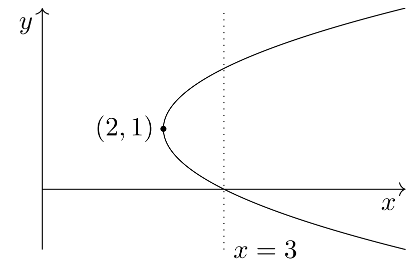
So, projection was at $\theta = \arctan 2$, as required.

3839. The numerator is a difference of two squares:

$$\begin{aligned} &\lim_{x\to 0} \frac{16^x - 1}{4^x - 1} \\ =\ &\lim_{x\to 0} \frac{(4^x+1)(4^x-1)}{4^x-1} \\ =\ &\lim_{x\to 0} 4^x + 1 \\ =\ &2. \end{aligned}$$

3840. (a) Completing the square, $x = (y-1)^2 + 2$. So, the coordinates of the vertex are $(2, 1)$.

(b) This is a positive parabola, x in terms of y, with vertex at $(2, 1)$:



(c) Solving for intersections with $x = 3$ (shown dotted above), we get $y = 0, 2$. So, the area of the region enclosed by the parabola and the line $x = 3$ is given by

$$\int_0^2 3 - (y^2 - 2y + 3)\, dy = \tfrac{4}{3}.$$

3841. Without loss of generality, we can scale and rotate the circles such that one is a unit circle centred at the origin, and the centre of the other lies on the x axis. The equations are $x^2 + y^2 = 1$ and $(x-k)^2 + y^2 = r^2$. For intersections,

$$(x-k)^2 - x^2 = r^2 - 1.$$

If $k = 0$, then $r = 1$ and the circles are the same, giving infinitely many points of intersection. If $k \neq 0$, then the equation is a quadratic, which can have a maximum of one double root, hence a maximum of one point of tangency. □

— Alternative Method —

Assume that two circles are tangent at distinct points A and B. At A, the tangent line is shared, so the radii lie along the same perpendicular line L_A. At B, the same is true.

① If AB is a diameter of one circle, then L_A and L_B are the same line, which must therefore be a diameter of both.

② If AB is not a diameter, then both centres are at the intersection of L_A and L_B.

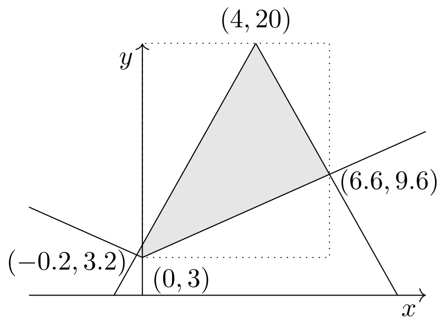
In both cases, the circles are identical, and tangent to each other at infinitely many points. □

3842. If an iteration $x_{n+1} = \text{g}(x_n)$ is to produce period 2 from $x_0 = \alpha$, then α must be a root of the equation $x = \text{g}^2(x)$. So, we solve

$$\begin{aligned} & x = 1 - \frac{1 - \frac{x}{2+x}}{2 + 1 - \frac{x}{2+x}} \\ \Longrightarrow\ & x = 1 - \frac{(2+x) - x}{3(2+x) - x} \\ \Longrightarrow\ & x = 1 - \frac{1}{3+x} \\ \Longrightarrow\ & x^2 + 2x + 2 = 0. \end{aligned}$$

This has $\Delta = -4 < 0$, so has no roots. Hence, no starting value x_0 will produce period 2.

3843. The boundary of the region is a quadrilateral:



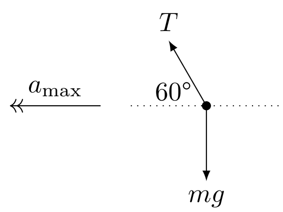
The coordinates of the vertices are as above. So, the dotted rectangle has area $6.6 \times 17 = 112.2$. To this we add the tiny triangle on the left, of area 0.1, and subtract three right-angled triangles, the sum of whose areas is

$$\tfrac{1}{2} \cdot 4 \cdot 16 + \tfrac{1}{2} \cdot 6.6 \cdot 6.6 + \tfrac{1}{2} \cdot 2.6 \cdot 10.4 = 67.3.$$

So, the area enclosed is $112.2 + 0.1 - 67.3 = 45$.

3844. (a) Since the sphere is smooth, the wall cannot exert friction on it. So, the only force which can exert a moment around the centre of the sphere is the tension. But the sphere does not rotate, so this force must exert no moment. Hence, its line of action must pass through the centre of the sphere.

(b) If the acceleration is below the critical value, then the reaction at the wall will be positive. If the acceleration is above the critical value, then the reaction at the wall will be negative, implying that the sphere has in fact left the wall. So, in the boundary case, the reaction at the wall is zero. The angle of inclination of the string is 60°, so the forces on the sphere are



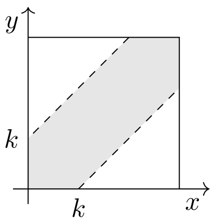
Resolving vertically,

$$\begin{aligned} T\sin 60^\circ &= mg \\ \Longrightarrow T &= \tfrac{2}{\sqrt{3}}mg. \end{aligned}$$

Then, resolving horizontally,

$$\begin{aligned} T\cos 60^\circ &= ma \\ \therefore\quad a_{\max} &= \tfrac{1}{\sqrt{3}}g \\ &= 5.66 \text{ ms}^{-2} \text{ (3sf)}. \end{aligned}$$

3845. Consider the unit square in the (x, y) plane as the possibility space. The successful region is shaded below, bounded by the lines $x - y = \pm k$:



The area of each unshaded triangle is $\frac{1}{2}(1-k)^2$. So, the area of the successful region is $1-(1-k)^2$, which is $2k - k^2$. Since the possibility space has area 1, this area is the probability:

$$\mathbb{P}\big(|x-y| < k\big) = 2k - k^2, \text{ as required.}$$

3846. (a) The first derivatives are $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = -2\sin 2t$. So,

$$\begin{aligned} \frac{dy}{dx} &= \frac{-2\sin 2t}{-\sin t} \\ &\equiv \frac{4\sin t\cos t}{\sin t} \\ &\equiv 4\cos t. \end{aligned}$$

Then, $\frac{d}{dt}(4\cos t) = -4\sin t$. So,

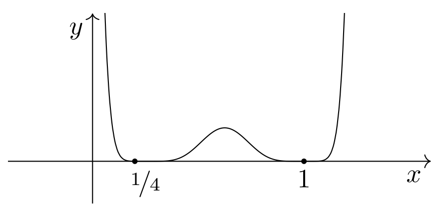
$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{-4\sin t}{-\sin t} \\ &\equiv 4, \text{ constant as required.} \end{aligned}$$

(b) Any curve with a constant, non-zero second derivative must be (part of) a parabola.

3847. The quadratic factorises as $(4x-1)(x-1)$, so the graph is

$$y = (4x-1)^6(x-1)^6.$$

This is a positive polynomial of degree 12, which has sextuple roots at $x = 1/4$ and $x = 1$. At such sextuple roots, the curve is tangent to the x axis, and doesn't cross it. The graph is as follows:



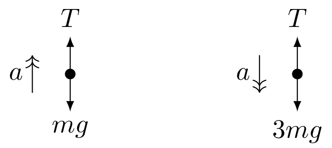
3848. Expanding the summand, we split the sum up and take out constant factors:

$$\begin{aligned} &\sum_i (x_i - \bar{x})^3 \\ \equiv\ &\sum_i \left(x_i^3 - 3x_i^2\bar{x} + 3x\bar{x}^2 - \bar{x}^3\right) \\ \equiv\ &\sum_i x_i^3 - 3\bar{x}\sum_i x_i^2 + 3\bar{x}^2\sum_i x - \bar{x}^3\sum_i 1. \end{aligned}$$

The third term is simplified by $\sum x \equiv n\bar{x}$. The sum in the fourth term is n copies of 1. This gives

$$\begin{aligned} &\sum_i x_i^3 - 3\bar{x}\sum_i x_i^2 + 3n\bar{x}^3 - n\bar{x}^3 \\ \equiv\ &\sum_i x_i^3 - 3\bar{x}\sum_i x_i^2 + 2n\bar{x}^3, \text{ as required.} \end{aligned}$$

3849. (a) The force diagrams are



Resolving along the cord, the tensions cancel, giving $2mg = 4a$, so $a = \frac{1}{2}$g. After 1 second, the displacements are $s = \pm\frac{1}{2}at^2 = \pm 2.5$ m, and the velocities are $v = at = \pm 10$ ms^{-1}.

(b) Since, as projectiles, the bobs have the same acceleration, we can ignore it. At $t = 1$, the lighter bob is 5 metres above the heavier, with relative speed 20 ms^{-1}.

The relative displacement needs to increase by 5 metres to make the rope taut. So, we need $\Delta t \times 20 = 5$, which is $\Delta t = 0.25$ s. Hence, a total time of 1.25 seconds has elapsed when the rope next becomes taut.

3850. Solving for intersections,

$$\begin{aligned} &(2x-1)(2y-1) + 4x^2y^2 = 0 \\ \therefore\quad &(2x-1)(-2x-1) + 4x^2(-x)^2 = 0 \\ \Longrightarrow\ &(2x^2-1)^2 = 0. \end{aligned}$$

Since the bracket $(2x^2 - 1)$ has single roots at $\pm\sqrt{2}/2$, the equation has double roots at $x = \pm\sqrt{2}/2$. These are points of tangency, as required.

3851. The equation of the tangent to $y = \sin x$, where x is measured in radians, is $y = x$.

The transformation that takes $y = \sin(x \text{ rad})$ to $y = \sin(x^\circ)$ is a stretch by scale factor $\frac{180}{\pi}$ in the x direction. Applying this transformation to the line $y = x$, the tangent is $y = \frac{\pi}{180}x$.

3852. The sine function has period 2π. So, the x and y components have t-periods $\frac{2\pi}{k_i}$. The combined period is the lcm of the individual periods.

(a) The components have periods π and ${}^{2\pi}/_{3}$, so the curve has period $\text{lcm}(\pi, {}^{2\pi}/_{3}) = 2\pi$.

(b) The components have periods ${}^{\pi}/_{4}$ and ${}^{\pi}/_{6}$, so the curve has period $\text{lcm}({}^{\pi}/_{4}, {}^{\pi}/_{6}) = {}^{\pi}/_{2}$.

3853. Let $u = \sqrt{x}$. Then $du = \frac{1}{2}x^{-\frac{1}{2}}\, dx$. We rearrange to $2\, du = \frac{1}{\sqrt{x}}\, dx$, since the RHS appears explicitly in the integral. The u limits are π and 2π.

Enacting the substitution,

$$\begin{aligned}&\int_{\pi^2}^{4\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx\\ &= \int_{\pi}^{2\pi} 2\cos u\, du\\ &= \Big[2\sin u\Big]_{\pi}^{2\pi}\\ &= 0, \text{ as required.}\end{aligned}$$

—— Alternative Method ——

To integrate by inspection, we notice that $\frac{1}{\sqrt{x}}$ is twice the derivative of $\sqrt{x}$. So,

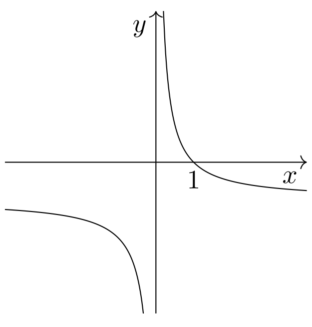
$$\begin{aligned}&\int_{\pi^2}^{4\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx\\ &= 2\int_{\pi^2}^{4\pi^2} \cos\sqrt{x}\cdot\big(\sqrt{x}\big)'\, dx\\ &= 2\Big[\sin\sqrt{x}\Big]_{\pi^2}^{4\pi^2}\\ &= 0, \text{ as required.}\end{aligned}$$

3854. Exponentiating and using the definition of a log, $a^{\text{LHS}} = xy$. Also

$$\begin{aligned}a^{\text{RHS}} &= a^{\log_a x + \log_a y}\\ &\equiv a^{\log_a x} a^{\log_a y}\\ &\equiv xy.\end{aligned}$$

This proves the log rule.

3855. This is true for a polynomial function f. It is not true, however, for e.g. $\text{f}(x) = \frac{1}{x} - 1$. This has a root at $x = 1$. However, as can be seen from the graph, any negative starting value will generate divergence to negative infinity:



3856. (a) The equation of line L is $y = m(x-1)+1$. Substituting this into the curve,

$$\begin{aligned}&m(x-1)+1 = x^2 + \frac{x-1}{x+1}\\ \Longrightarrow\ &mx^2 - m + x + 1 = x^3 + x^2 + x - 1\\ \Longrightarrow\ &x^3 + (1-m)x^2 - 2 + m = 0.\end{aligned}$$

(b) We know that $x = 1$ is a root of the above. Taking out a factor of $(x-1)$,

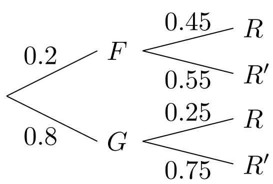
$$(x-1)\big(x^2 + (2-m)x + 2 - m\big) = 0.$$

For L and C to be tangent, we require the quadratic to have $\Delta = 0$. So,

$$\begin{aligned}&(2-m)^2 - 4(2-m) = 0\\ \Longrightarrow\ &(2-m)\big((2-m)-4\big) = 0\\ \Longrightarrow\ &m = \pm 2.\end{aligned}$$

This gives the two possible equations of L as $y = -2x + 3$ and $y = 2x - 1$.

3857. The possibility space, conditioned on whether a claim is genuine G or fraudulent F, is



Restricting the possibility space to rejected claims,

$$\begin{aligned}\mathbb{P}(G \mid R) &= \frac{\mathbb{P}(G\cap R)}{\mathbb{P}(R)}\\ &= \frac{0.8\times 0.25}{0.2\times 0.45 + 0.8\times 0.25}\\ &= \tfrac{20}{29}.\end{aligned}$$

3858. (a) Using x as the limit of integration (as in the diagram), let p be the variable of integration in the x direction. The cross-sectional area is given by

$$\begin{aligned}&\int_{-x}^{x} x^2 - p^2\, dp\\ &= \Big[x^2p - \tfrac{1}{3}p^3\Big]_{-x}^{x}\\ &= \tfrac{4}{3}x^3.\end{aligned}$$

Multiplying by the length of ditch l, which is assumed constant, $V \propto x^3$.

(b) The surface of the water is a $2x \times l$ rectangle. So, the surface area is proportional to x, which is proportional to the cube root of V. So,

$$\begin{aligned} \frac{dV}{dt} &= kV^{\frac{1}{3}} \\ \Longrightarrow \int V^{-\frac{1}{3}}\,dV &= \int k\,dt \\ \Longrightarrow \tfrac{3}{2}V^{\frac{2}{3}} &= kt + c. \end{aligned}$$

And we know that $V \propto x^3 = y^{\frac{3}{2}}$, so $V^{\frac{2}{3}} \propto y$. Hence, $y = at + b$, so the depth of water y falls at a constant rate.

3859. The double-angle formula $\sin 2\theta \equiv 2\sin\theta\cos\theta$ tells us that

$$\sin 4\theta \equiv 2\sin 2\theta \cos 2\theta.$$

With this, we can then factorise:

$$\begin{aligned} &2\sin 2\theta\cos 2\theta - 2\sin 2\theta + \cos 2\theta - 1 = 0 \\ \Longrightarrow\ &(2\sin 2\theta + 1)(\cos 2\theta - 1) = 0 \\ \Longrightarrow\ &\sin 2\theta = -\tfrac{1}{2} \text{ or } \cos 2\theta = 1. \end{aligned}$$

The first equation gives

$$\begin{aligned} 2\theta &= -\tfrac{5\pi}{6}, -\tfrac{\pi}{6}, \tfrac{7\pi}{6}, \tfrac{11\pi}{6} \\ \Longrightarrow \theta &= -\tfrac{5\pi}{12}, -\tfrac{\pi}{12}, \tfrac{7\pi}{12}, \tfrac{11\pi}{12}. \end{aligned}$$

The second equation gives

$$\begin{aligned} 2\theta &= -2\pi, 0, 2\pi \\ \Longrightarrow \theta &= -\pi, 0, \pi. \end{aligned}$$

So, the full solution is

$$x \in \left\{-\pi, -\tfrac{5\pi}{12}, -\tfrac{\pi}{12}, 0, \tfrac{7\pi}{12}, \tfrac{11\pi}{12}, \pi\right\}.$$

3860. Assume, for a contradiction, that an equilateral triangle has integer values P and A for perimeter and area. Then its side length is $l = \frac{P}{3}$, and the area is given by

$$\begin{aligned} A &= \frac{\sqrt{3}}{4}\left(\frac{P}{3}\right)^3 \\ &\equiv \frac{\sqrt{3}P^3}{108}. \end{aligned}$$

Hence, $108A = \sqrt{3}P^3$. The LHS is an integer, and the RHS is irrational. This is a contradiction. So, there is no equilateral triangle with integer values for both its perimeter and area. □

3861. (a) Rearranging to make y the subject,

$$\begin{aligned} y &= 1 - 2x^3 + x^6 \\ \Longrightarrow \frac{dy}{dx} &= -6x^2 + 6x^5 \\ \Longrightarrow \frac{d^2y}{dx^2} &= -12x + 30x^4. \end{aligned}$$

Solving for SPs,

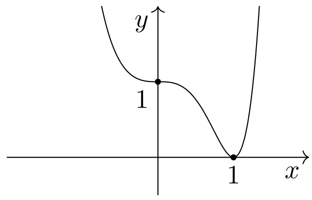
$$\begin{aligned} -6x^2 + 6x^5 &= 0 \\ \Longrightarrow x &= 0, 1. \end{aligned}$$

Evaluating the second derivatives at these points, we get 0 and 18. The latter value is conclusive, giving $(1, 0)$ as a local minimum. The former value needs further investigation. Factorising the second derivative,

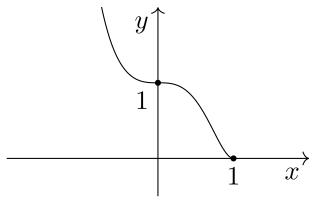
$$\frac{d^2y}{dx^2} = x\big(-12 + 30x^3\big).$$

This has a single root at $x = 0$. Hence, the second derivative changes sign. So, $(0, 1)$ is a point of inflection.

(b) The rearranged curve is a positive sextic with SPs as above:



The original curve, however, has no points where $1 - x^3 < 0$, which is $x > 1$. So, the original graph is



— Nota Bene —

Having removed points from the graph, the SP at $(0, 1)$ is only a one-sided minimum. This doesn't, in the final graph, distinguish it from a point of inflection. But the classification is still relevant to the sketch, as it distinguishes between a minimum and a maximum.

3862. Using the product rule, the derivatives are

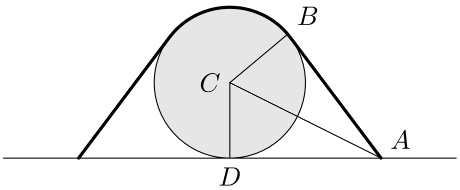
$$\begin{aligned} \frac{dx}{dt} &= Ae^t\cos t - Ae^t\sin t = x - y, \\ \frac{dy}{dt} &= Ae^t\sin t + Ae^t\cos t = x + y. \end{aligned}$$

Hence, the curve satisfies the DEs.

3863. To change a minimum at (p, q) into a maximum at $(-p, -q)$, we need to reflect the curve in both the x and y axes. So, we replace x by $-x$ and y by $-y$. This gives

$$\begin{aligned} -y &= (-x)^2 + a(-x) + b \\ \Longrightarrow y &= -x^2 + ax - b. \end{aligned}$$

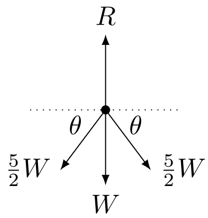
3864. (a) Drawing in a radius to produce $\triangle ABC$:



$\triangle ABC$ and $\triangle ACD$ are congruent. Each has angle α at A, where, calculated from $\triangle ACD$, $\sin\alpha = 1/\sqrt{5}$ and $\cos\alpha = 2/\sqrt{5}$. This gives

$$\begin{aligned}\sin\theta &= \sin 2\alpha \\ &= 2\sin\alpha\cos\alpha \\ &= 2\cdot\tfrac{1}{\sqrt{5}}\cdot\tfrac{2}{\sqrt{5}} \\ &= \tfrac{4}{5}, \text{ as required.}\end{aligned}$$

(b) Modelling the barrel as a particle, the force diagram is



Resolving vertically, $R = 5W\sin\theta + W = 5W$.

3865. (a) The derivative is $f'(x) = 1 - \frac{6}{5}x^{-\frac{2}{5}} + \frac{1}{5}x^{-\frac{4}{5}}$. The second and third terms are undefined at $x = 0$, since their indices are negative: this involves division. Hence, $f'(0)$ is undefined.

(b) Firstly, we solve $f'(x) = 0$:

$$\begin{aligned}&1 - \tfrac{6}{5}x^{-\frac{2}{5}} + \tfrac{1}{5}x^{-\frac{4}{5}} = 0 \\ \Longrightarrow\ & 5x^{\frac{4}{5}} - 6x^{\frac{2}{5}} + 1 = 0 \\ \Longrightarrow\ & \big(5x^{\frac{2}{5}} - 1\big)\big(x^{\frac{2}{5}} - 1\big) = 0 \\ \Longrightarrow\ & x^{\frac{2}{5}} = \tfrac{1}{5}, 1 \\ \Longrightarrow\ & x = \pm\tfrac{1}{5}^{\frac{5}{2}}, \pm 1.\end{aligned}$$

Testing these, we find that $f(\pm 1) = 0$. Hence, these are the x values in question.

— Alternative Method —

Factorising the original function,

$$\begin{aligned}f(x) &= x - 2x^{\frac{3}{5}} + x^{\frac{1}{5}} \\ &\equiv x^{\frac{1}{5}}\big(x^{\frac{4}{5}} - 2x^{\frac{2}{5}} + 1\big) \\ &\equiv x^{\frac{1}{5}}\big(x^{\frac{2}{5}} - 1\big)^2.\end{aligned}$$

This has a squared factor of $(x^{\frac{2}{5}} - 1)$, giving double roots at $x = \pm 1$. At these, $y = f(x)$ is tangent to the x axis, so $f'(x) = f(x) = 0$.

3866. For the velocity to reverse, it must instantaneously become zero at the cusp. So, we look for $\dot{x} = \dot{y} = 0$. Beginning with $\dot{x} = 0$, for $t \in [0, 2\pi)$:

$$\begin{aligned}&\sin t\cos t - (1 - \cos t)\sin t = 0 \\ \Longrightarrow\ & \sin t(2\cos t - 1) = 0 \\ \Longrightarrow\ & \sin t = 0 \text{ or } \cos t = \tfrac{1}{2} \\ \therefore\ & t = 0, \tfrac{\pi}{3}, \pi, \tfrac{5\pi}{3}, 2\pi, ...\end{aligned}$$

Secondly, $\dot{y} = 0$, again for $t \in [0, 2\pi)$:

$$\begin{aligned}&\sin^2 t + (1 - \cos t)\cos t = 0 \\ \Longrightarrow\ & 1 + \cos t - 2\cos^2 t = 0 \\ \Longrightarrow\ & (1 + 2\cos t)(1 - \cos t) = 0 \\ \Longrightarrow\ & \cos t = 1, -\tfrac{1}{2} \\ \therefore\ & t = 0, \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, 2\pi, ...\end{aligned}$$

At $t = 0$, both components of velocity are zero. To check that the direction reverses, we note that both velocities contain a single factor of $(1 - \cos t)$, which changes sign at $t = 0$. Hence, $\dot{x}$ and $\dot{y}$ both reverse at $t = 0$. So, the origin is a cusp.

— Nota Bene —

The direction also reverses at $t = 2n\pi$, for $n \in \mathbb{Z}$. But, since sin and cos both have period 2π, these times all correspond to the origin. Hence, the *path* has a single (x, y) cusp, as required.

3867. Dividing by ab, we can rewrite the boundary:

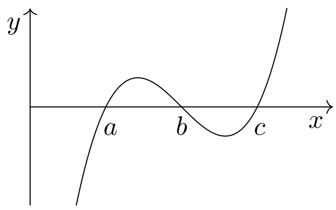
$$\begin{aligned}&\frac{x^2}{b} + \frac{y^2}{a} = 1 \\ \Longrightarrow\ & \left(\frac{x}{\sqrt{b}}\right)^2 + \left(\frac{y}{\sqrt{a}}\right)^2 = 1.\end{aligned}$$

This is a unit circle which has been transformed by a stretch scale factor $1/\sqrt{b}$ in the x direction and a stretch scale factor $1/\sqrt{a}$ in the y direction. Hence, the original area π is scaled by the product of these:

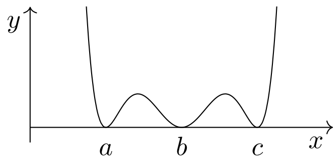
$$A_{\text{ellipse}} = \frac{\pi}{\sqrt{ab}}.$$

3868. Each graph has distinct roots at $x = a, b, c$, and is a positive polynomial.

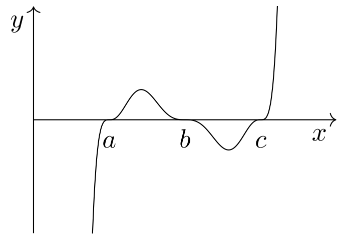
(a) A cubic with single roots:



(b) A sextic with double roots:



(c) A nonic with triple roots:



3869. Let $y = \arccos x$. So, $x = \cos y$. Differentiating implicitly with respect to x,

$$\begin{aligned} 1 &= -\sin y \frac{dy}{dx} \\ \implies \frac{dy}{dx} &= -\frac{1}{\sin y} \\ &= -\frac{1}{\sqrt{1-\cos^2 y}} \\ &= -\frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

This proves the result.

3870. By the binomial expansion, the expression $(1-x)^{\frac{1}{3}}$ can be approximated, for small x, by $1 - \frac{1}{3}x$. We substitute $x = 0.001$ into both:

$$\begin{aligned} & 0.999^{\frac{1}{3}} \approx 1 - \tfrac{1}{3} \cdot 0.001 \\ \implies & 0.027^{\frac{1}{3}} \cdot 37^{\frac{1}{3}} \approx 1 - \tfrac{1}{3} \cdot 0.001 \\ \implies & 0.3 \cdot \sqrt[3]{37} \approx \tfrac{2999}{3000} \\ \implies & \sqrt[3]{37} \approx \tfrac{2999}{900}, \text{ as required.} \end{aligned}$$

3871. (a) Differentiating with respect to t,

$$\begin{aligned} x &= A \ln t + B \\ \implies \frac{dx}{dt} &= \frac{A}{t} \\ \implies \frac{d^2x}{dt^2} &= -\frac{A}{t^2}. \end{aligned}$$

Substituting into the LHS,

$$\begin{aligned} & t\left(-\frac{A}{t^2}\right) + \frac{A}{t} \\ \equiv & -\frac{A}{t} + \frac{A}{t} \\ \equiv & \ 0. \end{aligned}$$

Therefore, for any values of the constants A and B, the proposed solution $x = A \ln t + B$ satisfies the DE.

(b) The velocity is

$$\frac{dx}{dt} = \frac{A}{t}.$$

So, the velocities at $t = 1$ and $t = 5$ are A and $\frac{A}{5}$. Hence, there is an 80% reduction in speed.

3872. Via a compound-angle identity, two double-angle identities and the first Pythagorean identity,

$$\begin{aligned} \sin 3t &\equiv \sin(2t + t) \\ &\equiv \sin 2t \cos t + \cos 2t \sin t \\ &\equiv 2 \sin t \cos^2 t + (1 - 2\sin^2 t) \sin t \\ &\equiv 2 \sin t (1 - \sin^2 t) + (1 - 2\sin^2 t) \sin t \\ &\equiv 3 \sin t - 4 \sin^3 t. \end{aligned}$$

So, the Cartesian equation is $x = 3y - 4y^3$.

3873. (a) We are told that $\mathbb{P}(B \mid A) = 4\,\mathbb{P}(A \mid B)$. So, using the conditional probability formula,

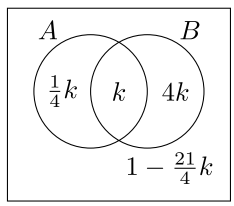
$$\begin{aligned} & \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = 4 \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\ \implies & \mathbb{P}(A \cap B)\,\mathbb{P}(B) = 4\,\mathbb{P}(A \cap B)\,\mathbb{P}(A). \end{aligned}$$

The conditional probabilities are non-zero, so we know that $\mathbb{P}(A \cap B) \neq 0$. Hence, we can divide through by $\mathbb{P}(A \cap B)$. This gives $\mathbb{P}(B) = 4\,\mathbb{P}(A)$, as required.

(b) We know that

$$\mathbb{P}(A) = \frac{k}{\frac{4}{5}} = \tfrac{5}{4}k.$$

Hence, $\mathbb{P}(A \cap B') = \frac{1}{4}k$. The same argument gives $\mathbb{P}(A' \cap B) = 4k$. Subtracting from 1, $\mathbb{P}(A' \cap B') = 1 - \frac{21}{4}k$. The Venn diagram is



(c) The union of A and B has probability $\frac{21}{4}k$. This is bounded by 0 below, and by 1 above. The lower bound isn't attainable, because the conditional probabilities would be undefined at $k = 0$. The upper bound is attainable; there is no such problem. So, $\mathbb{P}(A \cap B) \in (0, 4/21]$.

3874. The axis intercepts are $(0, 1)$ and $(1, 0)$. So, the line has equation $y = 1 - x$. For intersections, $1 - x = (1 - \sqrt{3})^3$. Let $z = \sqrt{x}$. So, we solve

$$\begin{aligned} & 1 - z^2 = (1 - z)^3 \\ \implies & 1 - z^2 = 1 - 3z + 3z^2 - z^3 \\ \implies & z^3 - 4z^2 + 3z = 0 \\ \implies & z(z-3)(z-1) = 0 \\ \implies & z = 0, 1, 3, \end{aligned}$$

The root we need is $z = 3$, so $x = 9$. This gives the third point of intersection as $(9, -8)$.

3875. (a) Since h is quartic, the graph $y = \mathrm{h}(x)$ cannot *cross* the x axis at α, or else it would have another root elsewhere. Hence, α must be a point of tangency with the x axis, that is, a repeated root.

(b) The Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{4x_n^4 - 4x_n^3 + 5x_n^2 - 4x_n + 1}{16x_n^3 - 12x_n^2 + 10x_n - 4}.$$

Running this iteration with $x_0 = 0$, we get $x_1 = \frac{1}{4}$, and then $x_n \to \frac{1}{2}$.

(c) The root $x = \frac{1}{2}$ is repeated, so $\mathrm{h}(x)$ must have a factor of $(2x-1)^2$. Taking this factor out,

$$\mathrm{h}(x) = (2x-1)^2(x^2+1).$$

The quadratic factor has no real roots, so we have factorised as far as is possible.

3876. We can write the first Pythagorean trig identity as

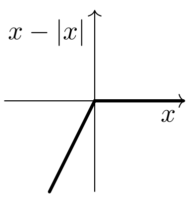
$$\sin^2 2x + \cos^2 2x \equiv 1.$$

Taking the positive square root because, for small x, $\cos 2x$ is positive, this gives

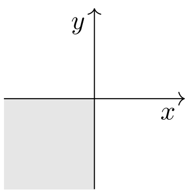
$$\cos 2x \approx \sqrt{1 - \sin^2 2x}.$$

Then, for small x in radians, $\sin 2x \approx 2x$. Hence, $\cos x \approx \sqrt{1-4x^2}$, as required.

3877. Consider the function $x \mapsto x - |x|$. Its graph is



So, neither of the factors $(x - |x|)$ and $(y - |y|)$ can be positive. Hence, their product is only positive when both are negative. So, the region is



The axes are excluded from the region.

3878. The relevant derivative (velocity in the x direction) is $\dot{x} = 3\cos 3t$. Setting this to zero for tangents parallel to the y axis,

$$\begin{aligned} 3t &= \tfrac{\pi}{2}, \tfrac{3\pi}{2}, ... \\ \Longrightarrow t &= \tfrac{\pi}{6}, \tfrac{3\pi}{6}, ... \end{aligned}$$

The t values continue in increments of $\frac{2\pi}{6}$. There are six points to find. At the first six t values, we calculate the coordinates to be as follows:

t	$\frac{\pi}{6}$	$\frac{3\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{9\pi}{6}$	$\frac{11\pi}{6}$
x	1	-1	1	-1	1	-1
y	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$

3879. The curves are reflections of each other in the line $y = x$. So, if the curves intersect with $y = x$, then they intersect with each other. The equation $\mathrm{f}(x) = x$ is a cubic equation, which must have at least one root. Hence, $y = \mathrm{f}(x)$ must intersect $y = x$, and therefore $x = \mathrm{f}(y)$. □

3880. Let $u_1 = a$ and $u_2 = b$. In terms of a and b, the sequence continues

$$\begin{aligned} u_3 &= b - a, \\ u_4 &= (b-a) - b \equiv -a, \\ u_5 &= (-a) - (b-a) \equiv -b, \\ u_6 &= (-b) - (-a) \equiv a - b, \\ u_7 &= (a-b) - (-b) \equiv a, \\ u_8 &= (a) - (a-b) \equiv b. \end{aligned}$$

Since $u_7 = a = u_1$ and $u_8 = b = u_2$, the sequence is periodic, as required.

3881. Writing the integrand in partial fractions,

$$\begin{aligned} &\int_0^3 \frac{3x+1}{(x+1)(x+3)}\,dx \\ &= \int_0^3 \frac{4}{x+3} - \frac{1}{x+1}\,dx \\ &= \Big[4\ln|x+3| - \ln|x+1|\Big]_0^3 \\ &= (4\ln 6 - \ln 4) - (4\ln 3 - \ln 1). \end{aligned}$$

Using log rules, this is

$$\begin{aligned} &\ln \tfrac{6^4}{4\cdot 3^4} \\ &= \ln 4, \text{ as required.} \end{aligned}$$

3882. (a) Setting $a = b = 1$,

$$\begin{aligned} &\cos y + \cos x = 0 \\ \Longrightarrow\ &\cos y = -\cos x \\ \Longrightarrow\ &\cos y = \cos(\pi - x). \end{aligned}$$

The first two solution lines are $y = \pi - x$ and $y = 2\pi - (\pi - x) = \pi + x$. Since cos is even, $y = -\pi + x$ and $y = -\pi - x$ are also solution lines. These four form a square centred on the origin, with side length $\sqrt{2}\pi$ and area $2\pi^2$.

(b) Replacing x by ax stretches the regions by scale factor $1/a$ in the x direction. Replacing y by by stretches the regions by scale factor $1/b$ in the y direction. Hence, the area in the general case is

$$A = \frac{2\pi^2}{ab}.$$

3883. The LHS is $e^{2x}+e^{-2x}$. The RHS is

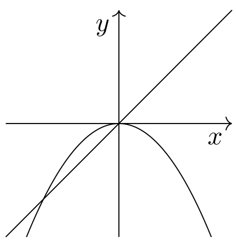
$$\begin{aligned}&\left(e^x+e^{-x}\right)^2\\ \equiv\ &e^{2x}+2e^x\cdot e^{-x}+e^{-2x}\\ \equiv\ &e^{2x}+2+e^{-2x}.\end{aligned}$$

Hence, with $k=-2$, the identity holds.

3884. Factorising the LHS,

$$\begin{aligned}&x^3-x^2y+2xy-2y^2\\ \equiv\ &(x-y)(x^2+2y).\end{aligned}$$

So, the (x,y) points which satisfy the relation are either on the line $x-y=0$ or on the parabola $x^2+2y=0$:



3885. The first student is wrong. Choosing $u=x^2$ means $\frac{dv}{dx}=\ln x$. So $\frac{du}{dx}=2x$. And v is then the integral of $\ln x$, which is $x\ln x-x$. So, the integral is

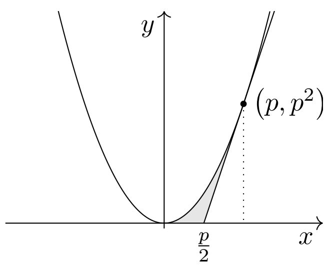
$$x^2(x\ln x-x)-\int 2x(x\ln x-x)\,dx.$$

The integral is now even more complicated than it was to start with.

The right choice is $u=\ln x$ and $\frac{dv}{dx}=x^2$, so that $\frac{du}{dx}=\frac{1}{x}$ and $v=\frac{1}{3}x^3$. This way, the integral is

$$\begin{aligned}&\tfrac{1}{3}x^3\ln x-\int\tfrac{1}{3}x^2\,dx\\ =\ &\tfrac{1}{3}x^3\ln x-\tfrac{1}{9}x^3+c.\end{aligned}$$

3886. The equation of the tangent line is $y=2px-p^2$. This intersects the x axis at $x=\frac{p}{2}$. So, the scenario is



The area of the triangle below the tangent line is $\frac{1}{4}p^3$. So, the shaded area is given by

$$\begin{aligned}A&=\int_0^p x^2\,dx-\tfrac{1}{4}p^3\\ &\equiv\left[\tfrac{1}{3}x^3\right]_0^p-\tfrac{1}{4}p^3\\ &\equiv\tfrac{1}{3}p^3-\tfrac{1}{4}p^3\\ &\equiv\tfrac{1}{12}p^3,\ \text{as required.}\end{aligned}$$

3887. Using two log rules, we can rewrite the middle term as follows:

$$\log_4 x^6\equiv\log_2 x^3\equiv 3\log_2 x.$$

This gives a quadratic in $\log_2 x$:

$$\begin{aligned}&(\log_2 x)^2-3\log_2 x-4=0\\ \Longrightarrow\ &(\log_2 x+1)(\log_2 x-4)=0\\ \Longrightarrow\ &\log_2 x=-1,4\\ \Longrightarrow\ &x=\tfrac{1}{2},16.\end{aligned}$$

3888. The graph is polynomial, so the only symmetries it could possibly have are as follows:

① rotational symmetry of order 2,
② reflective symmetry in a line $x=k$.

The graph is quartic, so cannot have rotational symmetry. This rules out ①. Factorising, we have $y=x^3(x+1)$. This has a triple root at $x=0$ and a single root at $x=-1$. These cannot be images of each other under reflection in $x=k$, because the gradient is 0 at $x=0$, but non-zero at $x=-1$. This rules out ②.

So, the graph has no symmetry.

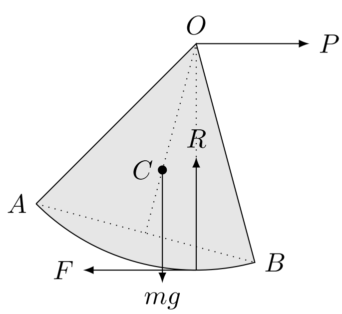
3889. Each of the curves is periodic, period π. Solving for intersections in $[0,\pi)$,

$$\begin{aligned}&\sqrt{3}\sec^2 x=4\tan x\\ \Longrightarrow\ &\sqrt{3}(1+\tan^2 x)=4\tan x\\ \Longrightarrow\ &\sqrt{3}\tan^2 x-4\tan x+\sqrt{3}=0\\ \Longrightarrow\ &\tan x=\tfrac{4\pm\sqrt{16-12}}{2\sqrt{3}}\\ &\qquad=\tfrac{3}{\sqrt{3}},\tfrac{1}{\sqrt{3}}\\ \therefore\ &\ x=\tfrac{\pi}{6},\tfrac{\pi}{3}.\end{aligned}$$

So, the area of the regions enclosed is

$$\begin{aligned}A&=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}4\tan x-\sqrt{3}\sec^2 x\,dx\\ &=\Big[4\ln|\sec x|-\sqrt{3}\tan x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}\\ &=(\ln 16-3)-(\ln\tfrac{16}{9}-1)\\ &=\ln 9-2,\ \text{as required.}\end{aligned}$$

3890. (a) The force diagram is as follows. The ground must be rough, or else the toy would accelerate to the right.



(b) Horizontally, $F = P$. Now, call the radius of the sector 1, and take moments about O.

For the moment of the weight, we need the distance between its line of action and O. The height of the equilateral triangle OAB is $\sqrt{3}/2$. So, $|OC| = \sqrt{3}/3$. Hence, the distance from the line of action of the weight to O is $|OC|\sin\theta$. So, the moment of the weight is

$$\tfrac{\sqrt{3}}{3}mg\sin\theta.$$

The reaction force has no moment about O. The moment of the friction, since the sector has radius 1, is F, which is equal to P. This gives, as required,

$$P = \tfrac{\sqrt{3}}{3}mg\sin\theta.$$

3891. The derivative f′ is quadratic with roots at $x = \pm 2$. So, it must have the following form, for $k \neq 0$:

$$\begin{aligned} f'(x) &= k(x+2)(x-2) \\ &\equiv kx^2 - 4k. \end{aligned}$$

This gives $f''(x) = 2kx$, which is zero at $x = 0$, and also changes sign. So, $y = f(x)$ has a point of inflection at $x = 0$, as required.

3892. (a) $\mathbb{P}(A' \cup B') \equiv 1 - \mathbb{P}(A \cap B) = 1 - y$.

(b) Since the intersection is contained in the union, $\mathbb{P}(A \cap B \mid A \cup B) = y/x$.

(c) Successful outcomes here are those in exactly one of A or B. This is $x - y$.

3893. For intersections,

$$x^{\frac{1}{n}} = x^{\frac{1}{n-1}}.$$

Raising both sides to the power $n(n-1)$,

$$\begin{aligned} & x^{n-1} = x^n \\ \Longrightarrow\ & x^{n-1}(1-x) = 0 \\ \Longrightarrow\ & x = 0, 1. \end{aligned}$$

Over the domain $(0, 1)$, the graph $y = x^n$ is above the graph $y = x^{n-1}$. So, the area is given by

$$\begin{aligned} A &= \int_0^1 x^{\frac{1}{n}} - x^{\frac{1}{n-1}}\,dx \\ &\equiv \left[\frac{n}{n+1}x^{\frac{n+1}{n}} - \frac{n-1}{n}x^{\frac{n}{n-1}}\right]_0^1 \\ &\equiv \frac{n}{n+1} - \frac{n-1}{n} \\ &\equiv \frac{n^2 - (n+1)(n-1)}{n(n+1)} \\ &\equiv \frac{1}{n^2+n}, \text{ as required.} \end{aligned}$$

3894. (a) Using $\sin 2x \equiv 2\sin x\cos x$, we then divide top and bottom by $\cos^2 x$:

$$\begin{aligned} & \frac{1-\sin 2x}{1+\sin 2x} \\ \equiv\ & \frac{1-2\sin x\cos x}{1+2\sin x\cos x} \\ \equiv\ & \frac{\sec^2 x - 2\tan x}{\sec^2 x + 2\tan x}. \end{aligned}$$

(b) Using the second Pythagorean trig identity, we can rewrite the above, and then factorise the resulting quadratics in $\tan x$:

$$\begin{aligned} & \frac{1+\tan^2 x - 2\tan x}{1+\tan^2 x + 2\tan x} \\ \equiv\ & \frac{(1-\tan x)^2}{(1+\tan x)^2}. \end{aligned}$$

This tells us that

$$\frac{1-\sin 2x}{1+\sin 2x} \equiv \left(\frac{1-\tan x}{1+\tan x}\right)^2.$$

Both sides of this are non-negative. So, when we take the positive square root of the LHS, we must ensure that the RHS is also positive. This requires a mod function, which gives the required result:

$$\sqrt{\frac{1-\sin 2x}{1+\sin 2x}} \equiv \left|\frac{1-\tan x}{1+\tan x}\right|.$$

3895. We are told that

$$\begin{aligned} 310 &= \tfrac{n}{2}\big(8 + (n-1)d\big), \\ 374 &= \tfrac{n+1}{2}(8 + nd). \end{aligned}$$

Simplifying these,

$$\begin{aligned} 620 &= 8n + n(n-1)d, \\ 748 &= 8(n+1) + n(n+1)d. \end{aligned}$$

We subtract $(n+1)$ copies of the first equation from $(n-1)$ copies of the second, giving

$$\begin{aligned} & 748(n-1) - 620(n+1) \\ & \qquad = 8(n+1)(n-1) - 8n(n+1) \\ \Longrightarrow\ & n = 10. \end{aligned}$$

This gives $d = 6$. Looking for a partial sum with value 1000,

$$\begin{aligned} & \tfrac{n}{2}\big(8 + 6(n-1)\big) = 1000 \\ \Longrightarrow\ & n = -18.4, 18.1. \end{aligned}$$

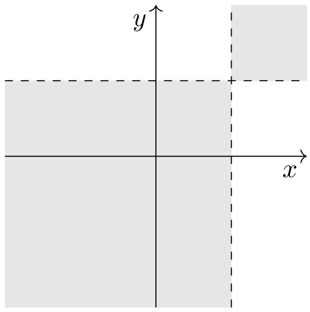
Rejecting the negative root, we take $n = 19$, so the value of the first partial sum to exceed 1000 is

$$\tfrac{19}{2}\big(8 + 6(19-1)\big) = 1102.$$

3896. Rearranging, we factorise:

$$\begin{aligned} & xy - x - y + 1 > 0 \\ \implies & (x-1)(y-1) > 0. \end{aligned}$$

For the product to be positive, either $(x-1)$ and $(y-1)$ must both be positive or both be negative. This gives two regions of the (x, y) plane, bounded by the lines $x = 1$ and $y = 1$:



3897. Spotting the root $x = -1$, we take out $(x+1)$. This leaves $x^4 + x^2 + 1$. The quadratic factors must be monic, and the constant terms must multiply to 1, so we are looking for

$$x^4 + x^2 + 1 \equiv (x^2 + ax + 1)(x^2 + bx + 1).$$

Equating coefficients of x^3, we require $a = -b$. Equating coefficients of x^2, we need $2 + ab = 1$. Solving these simultaneously gives 1 and -1. So,

$$\begin{aligned} & 1 + x + x^2 + x^3 + x^4 + x^5 \\ \equiv\ & (x+1)(x^2 + x + 1)(x^2 - x + 1). \end{aligned}$$

— Nota Bene —

Both quadratics have $\Delta = -3$, so are irreducible.

3898. Rewriting as the sum of a polynomial and a proper fraction,

$$\begin{aligned} & \frac{2x^5 + 5x^2}{x^3 + 1} \\ \equiv\ & \frac{2x^2(x^3+1) + 3x^2}{x^3 + 1} \\ \equiv\ & 2x^2 + \frac{3x^2}{x^3 + 1}. \end{aligned}$$

The numerator of the fraction is the derivative of its denominator, so we integrate by inspection:

$$\begin{aligned} & \int 2x^2 + \frac{3x^2}{x^3+1}\, dx \\ =\ & \tfrac{2}{3}x^3 + \ln|x^3 + 1| + c. \end{aligned}$$

3899. In order to be able to construct a quadrilateral, it is sufficient that each length is individually smaller than the sum of the others. We can write the AP

$$\{a, a+d, a+2d, a+3d\}.$$

Wlog, we can assume that $d \geq 0$ (if not, we could relabel the points in reverse order), which gives the sum of the three smallest terms as

$$a + (a+d) + (a+2d) = 3a + 3d.$$

Since $a > 0$, this is strictly greater than $a + 3d$. Hence, the longest side can be used to close the quadrilateral. QED.

3900. The possible outcomes for the sum on two dice are $\{2, 3, ..., 12\}$. These have probabilities

$$\left\{\tfrac{1}{36}, \tfrac{2}{36}, ..., \tfrac{5}{36}, \tfrac{6}{36}, \tfrac{5}{36}, ..., \tfrac{2}{36}, \tfrac{1}{36}\right\}.$$

So, the probability p that the first two give the same total score as the last two is the sum of the squares of these probabilities:

$$\begin{aligned} p &= 2\left(\tfrac{1}{36}^2 + \tfrac{2}{36}^2 + ... + \tfrac{5}{36}^2\right) + \tfrac{6}{36}^2 \\ &= \tfrac{73}{648}. \end{aligned}$$

— End of 39th Hundred —